The fundamental solution and the uniqueness theorem of Cauchy problem for the linear parabolic system

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1. Introduction. In this note we study the following parabolic system.

$$\frac{\partial u_i}{\partial t}-\sum_{j=1}^{N}\sum_{\Sigma k_s=2b}A_{ij}^{(k_1,\cdots,k_n)}(x_1,\cdots,x_n,t)\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}u_j$$

$$(1)\qquad -\sum_{j=1}^{N}\sum_{\Sigma k_s<2b}A_{ij}^{(k_1,\cdots,k_n)}(x_1,\cdots,x_n,t)\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}u_j=0,\ i=1,\cdots,N.$$

By the use of matrices we rewrite (1) as follows:

$$(1')\qquad LU=\frac{\partial U}{\partial t}-P_0\left(t,x,\frac{1}{2\pi i}\frac{\partial}{\partial x}\right)U-P_1\left(t,x,\frac{1}{2\pi i}\frac{\partial}{\partial x}\right)U=0$$

In [1], [2] S. D. Eidelman showed the existence of the fundamental solution and the solution of Cauchy problem for (1) under the conditions of a), b), c,) and proved the uniqueness of the solution of (1) under the conditions of a), b), d).

a) *The equation* $|P_0(t,x,\sigma)-\lambda E|=0$ *has the roots satisfying real part of* $\lambda<-\delta$, $t_0\leqq t\leqq T$, $-\infty<x_s<\infty$, $s=1,\cdot\cdot,n$, $|\sigma|=1$, $\delta>0$.

b) *The elements of* $P_0(t,x,s)$ *are continuous and bounded functions of* x,t. *The continuity with respect to* t *is uniform, and the elements of* P_0 *have continuous and bounded derivatives of order* $\leqq\gamma_1$ $(\gamma_1\geqq 2b+1)$ *with respect to* $x_1,\cdots,x_n$, $t_0\leqq t\leqq T$, $-\infty<x_s<\infty$, $s=1,\cdots,n$.

c) *The elements of* $P_1(t,x,s)$ *are continuous and bounded functions of* x,t. *Their derivatives of order* $\leqq\gamma_1-2b$ *with respect to* $x_1,\cdots,x_n$ *are continuous and bounded,* $t_0\leqq t\leqq T$, $-\infty<x_s<\infty$, $s=1,\cdots,n$.

d) *In c) we replace* γ_1-2b *by* $\sum k_s+1$.

Our objects are to establish the fundamental solution under the conditions of a), e), f), and to prove the uniqueness theorem for (1) under the conditions of a), e), g).

e) *In b) we replace* γ_1 *by* $2b$, *and the derivatives of order* $2b$ *satisty the condition L).*

f) *In c) we replace* γ_1-2b *by* 0, *and the elements of* P_1 *satisfy the condition L).*

g) *In d) we replace* $\sum k_s+1$ *by* $\sum k_s$, *and the dirivatives of order* $\sum k_s$ *satisfy the condition L).*

L) There exist positive constants K, R, γ *such that* $|f(x, t)-f(y, t)| \leqq K\sum_1^n |x_s-y_s|^\gamma$, $0<\gamma\leqq 1$ *for* $x_s-R\leqq y_s\leqq x_s+R$, $s=1, \cdots, n$, $-\infty<x_s<\infty$, $t_0\leqq t\leqq T$.

2. We go on studying of the theory alike in the case of [1].

Let $G(t, \tau, x-\xi, y)$ be Green's matrix of $\partial U/\partial t=P_0(t, y, 1/2\pi i\cdot\partial/\partial x)U$, then we have lemma 1 under the conditions of a), e).

LEMMA 1 (*by Eidelman.* [1] § 2.2)

$$|D_y^{m_1}D_x^{m_2}G(t, \tau, x-\xi, y)| \leqq \frac{E_{m_1,m_2}}{(t-\tau)^{(n+m_2)/2b}}\exp\Big(-c^*_{m_1m_2}\sum_1^n|x_s-\xi_s|^q(t-\tau)^{-1/(2b-1)}\Big),\ (m_1=0, \cdots, 2b),$$

where $t>\tau$, $\dfrac{1}{q}+\dfrac{1}{2b}=1$, $E_{m_1m_2}$, $c^*_{m_1m_2}$ *are positive constants.*

LEMMA 2. *If a continuous function* $\varphi(t, \tau, x, \xi)$ *satisfies the following inequalities*

$$|\varphi(t, \tau, x, \xi)|\leqq\frac{c_1}{(t-\tau)^{(n+2b-1)/2b}}\exp\Big(-c^{**}\sum_1^n|x_s-\xi_s|^q(t-\tau)^{-1/(2b-1)}\Big),$$

$$|\varphi(t, \tau, x+h, \xi)-\varphi(t, \tau, x, \xi)| \leqq\frac{ch^\gamma}{(t-\tau)^{(n+2b-1+\gamma)/2b}}\exp\Big(-c^{**}\sum_1^n|x_s-\xi_s|^q(t-\tau)^{-1/(2b-1)}\Big), \text{ where}$$

$0<h\leqq(t-\tau)^{1/(2b)}$, $0<\gamma\leqq 1$, $t>\tau$, c, c_1, c^{**} *are positive constants, we have*

$$L\Big(\int_\tau^t d\beta\int G(t, \beta, x-y, x)\varphi(\beta, \tau, y, \xi)dy\Big)=\varphi(t, \tau, x, \xi) +\int_\tau^t d\beta\int LG(t, \beta, x-y, x)\varphi(\beta, \tau, y, \xi)dy$$

under the conditions of a), e), f).

PROOF.

$$\begin{aligned} &D_x^{2b}\int_\tau^t d\beta\int G(t, \beta, x-y, x)\varphi(\beta, \tau, y, \xi)dy \\ &=\int_\tau^t d\beta\int D_x^{2b}G(t, \beta, x-y, x)\{\varphi(\beta, \tau, y, \xi)-\varphi(\beta, \tau, x, \xi) \\ &+\varphi(\beta, \tau, x, \xi)-\varphi(t, \tau, x, \xi)\}dy \\ &+\varphi(t, \tau, x, \xi)\int_\tau^t d\beta\int D_x^{2b}G(t, \beta, x-y, x)dy\,. \end{aligned} \tag{2}$$

$$\frac{\partial}{\partial t}\int_\tau^t d\beta\int G(t, \beta, x-y, x)\varphi(\beta, \tau\, y, \xi)dy=\varphi(t, \tau, x, \xi)+$$

$$(3)\qquad \int_\tau^t d\beta\int\frac{\partial}{\partial t}G(t,\beta,x-y,x)\{\varphi(\beta,\tau,y,\xi)-\varphi(\beta,\tau,x,\xi)$$
$$+\varphi(\beta,\tau,x,\xi)-\varphi(t,\tau,x,\xi)\}dy+\varphi(t,\tau,x,\xi)$$
$$\times\int_\tau^t d\beta\int\frac{\partial}{\partial t}G(t,\beta,x-y,x)dy\ .$$

By the aid of (2), (3) and the properties of Green's matrix we obtain lemma 2.

Let $\varphi(t,\tau,x,\xi)$ be a solution of

$$K(t,\tau,x-\xi,x)+\varphi(t,\tau,x,\xi)+\int_\tau^t d\beta\int K(t,\beta,x-y,x)\varphi(\beta,\tau,y,\xi)dy=0\ ,$$

where $K(t,\tau,x-\xi,x)=LG(t,\tau,x-\xi,x)$. By the use of lemma 2 we make the fundamental solution $Z(t,\tau,x,\xi)$ as follows:

$$Z(t,\tau,x,\xi)=G(t,\tau,x-\xi,x)+\int_\tau^t d\beta\int G(t,\beta,x-y,x)\varphi(\beta,\tau,y,\xi)dy\ ,$$

where $\varphi(t,\tau,x,\xi)=\sum_{m=0}^{\infty}\varphi_m(t,\tau,x,\xi)$,

$$\varphi_0(t,\tau,x,\xi)=-K(t,\tau,x-\xi,x)=K^*(t,\tau,x-\xi,x)\ ,$$
$$\varphi_m(t,\tau,x,\xi)=\int_\tau^t d\beta\int K^*(t,\beta,x-y,x)\varphi_{m-1}(\beta,\tau,y,\xi)dy\ .$$

Then we have lemma 3.

LEMMA 3. *If we assume the conditions of* a), e), f), $\varphi(t,\tau,x,\xi)$ *satisfies the inequalities of lemma* 2.

PROOF. It is stated in [1], § 2,4 that the function $\varphi(t,\tau,x,\xi)$ satisfies the first inequality, and

$$(4)\qquad \begin{aligned}&|\varphi_m(t,\tau,x,\xi)|\\ &\leqq\frac{A_m(\varepsilon)}{(t-\tau)^{(n+2b-m-1)/2b}}\exp\Bigl(-c^*(1-m\varepsilon)\sum_1^n|x_s-\xi_s|^q(t-\tau)^{-1/(2b-1)}\Bigr)\ .\\ &(m=0,1,\cdots,n+2b-1)\ ,\end{aligned}$$

$$(5)\qquad \begin{aligned}&|\varphi_{2b+n+k}(t,\tau,x,\xi)|\\ &\leqq A_0(c^*A)^{k+1}(t-\tau)^{(k+1)/2b}\prod_{s=0}^{k}B\Bigl(\frac{1}{2b},1+\frac{s}{2b}\Bigr)\\ &\times\exp\Bigl(-c^{**}\sum_1^n|x_s-\xi_s|^q(t-\tau)^{-1(/2b-1)}\Bigr)\ ,\\ &(k=0,1,2,\cdots,[1],\S\,2,4)\ ,\end{aligned}$$

where ε is any positive number, $c^*=\mathrm{Min}c^*_{m_1m_2}(m_1=0,\cdots,2b,\ m_2=0,\cdots,2b)$,

$c^{**} < c^*$.

To prove the second inequality, first we show that $K(t, \tau, x-\xi, x)$ satisfies it.

Considering $K=LG$, we see that $K(t, \tau, x-\xi, x)$ is finite sums of forms $a(x, t)K^\circ(t, \tau, x-\xi, x)$, where $a(x, t)$ is $A_{ij}^{(k_1, \cdots, k_n)}(x_1, \cdots, x_n, t)$,

$$K^\circ = D_x^m G(t, \tau, x-\xi, x) , \quad m=0, \cdots, 2b-1 .$$

$$\begin{aligned}&|a(x+h, t)K^\circ(t, \tau, x+h-\xi, x+h)-a(x, t)K^\circ(t, \tau, x-\xi, x)|\\ &\quad \leqq |a(x+h, t)K^\circ(t, \tau, x+h-\xi, x+h)-a(x, t)K^\circ(t, \tau, x+h-\xi, x+h)|\\ &\quad +|a(x, t)|\,|K^\circ(t, \tau, x+h-\xi, x+h)-K^\circ(t, \tau, x-\xi, x+h)|\\ &\quad +|a(x, t)|\,|K^\circ(t, \tau, x-\xi, x+h)-K^\circ(t, \tau, x-\xi, x)|=J_1+J_2+J_3 .\end{aligned}$$

Using the condition L) for the coefficients in J_1, lemma 1 in J_2, J_3, and applying the theorem of the mean, we have

$$\begin{aligned}I_0(h)&=|K(t, \tau, x+h-\xi, x+h)-K(t, \tau, x-\xi, x)|\\ &\leqq \frac{c_1 h^\gamma}{(t-\tau)^{(n+2b-1+\gamma)/2b}} \exp\left(-c^* \sum_1^n |x_s-\xi_s|^q (t-\tau)^{-1/(2b-1)}\right) ,\end{aligned}$$

where $0<h\leqq(t-\tau)^{1/(2b)}$, c_1 is a positive constant.

To prove the second inequality, writing $|\varphi_m(t, \tau, x+h, \xi)-\varphi_m(t, \tau, x, \xi)|= I_m(h)$, we denote the contribution to the integral $I_m(h)$ from the interval $\left[\tau, \tau+\frac{t-\tau}{2}\right]$ by $I_m^1(h)$, and from the interval $\left[\tau+\frac{t-\tau}{2}, t\right]$ by $I_m^2(h)$. It is to be noted that $I_0(h)$ can be used in $I_m^I(h)$, but not in $I_m^2(h)$.

i) By $I_0(h)$ and the inequality (5) we have

$$I_{2b+n+k+1}^I(h) \leqq h^\gamma c_1 c^* A (t-\tau)^{(1-\gamma)/2b} B\left(\frac{1-\gamma}{2b}, 1+\frac{k+1}{2b}\right)\theta_k' = \theta_{2b+n+k+1} ,$$

where θ_k' is the right term of the inequality (5). It shows $I_{2b+n+k+1}^I(h)$ satisfies the second inequality.

ii) We appreciate

$$\begin{aligned}I_{2b+n+k+1}^2(h)=\Bigg|\int_{\tau+(t-\tau)/2}^t d\beta \int \{&K^*(t, \beta, x+h-y, x+h)\varphi_{2b+n+k}(\beta, \tau, y, \xi)\\ &-K^*(t, \beta, x-y, x)\varphi_{2b+n+k}(\beta, \tau, y, \xi)\}dy\Bigg| .\end{aligned}$$

Changing the variable y by $x+h-\alpha$ in the first integrand and y by $x-\alpha$ in the second integrand, we have the following inequality

$$\begin{aligned}&I_{2b+n+k+1}^2(h)\\ &\leqq \left|\int_{\tau+(t-\tau)/2}^t d\beta \int \{K(t, \beta, \alpha, x+h)-K(t, \beta, \alpha, x)\}\varphi_{2b+n+k}(\beta, \tau, x+h-\alpha, \xi)\cdot d\alpha\right|\end{aligned}$$

$$+\left|\int_{\tau+(t-\tau)/2}^{t} d\beta\int K(t,\beta,\alpha,x)\{\varphi_{2b+n+k}(\beta,\tau,x+h-\alpha,\xi)\right.$$
$$\left.-\varphi_{2b+n+k}(\beta,\tau,x-\alpha,\xi)\}d\alpha\right| .$$

In the first integral using lemma 1, the condition L) for the coefficients and the theorem of the mean, we can appreciate it by the same value as i). By the use of mathematical induction, we can prove that the second integral is not larger than $3^{2b+n+k}\theta_{2b+n+k+1}$. In consequence of the above we conclude that $I_{2b+n+k+1}(h)\leqq 3^{2b+n+k+1}\theta_{2b+n+k+1}$, and $\sum_{k=0}^{\infty} I_{2b+n+k+1}(h)$ is uniformly convergent $(-\infty<x_s<\infty\, s=1,\cdots,n,\ t_0\leqq\tau<t\leqq T)$. We can appreciate $I_1,\cdots,I_{2b+n}$ in the same way as I_0, $I_{2b+n+k}(k>0)$[1]. Thus we complete the proof. By lemma 1, 2, 3 we obtain theorem 1.

THEOREM 1. *Under the conditions of* a), e), f) $Z(t,\tau,x,\xi)=G(t,\tau,x-\xi,x)+\int_\tau^t d\beta\int G(t,\beta,x-y,x)\varphi(\beta,\tau,y,\xi)dy$ *is the fundamental solution of* (1). *Denote* $\int_\tau^t d\beta\int G(t,\beta,x-y,x)\varphi(\beta,\tau,y,\xi)=W(t,\tau,x,\xi)$, *then*

$$|D_x^m W(t,\tau,x,\xi)|$$
$$\leqq\frac{E}{(t-\tau)^{(n+m-1)/2b}}\exp\left(-(c^{**}-\delta_0)\sum_1^n|x_s-\xi_s|^q(t-\tau)^{-1/(2b-1)}\right),$$
$$(m=0,\cdots,2b-1) .$$

$D_x^{2b}W$, D_tW *are finite for* $t_0\leqq\tau<t\leqq T$, *where* δ_0 *is any positive constant* $<c^{**}$. *The properties of* $G(t,\tau,x-\xi,x)$ *are seen in lemma* 1.

3. THEOREM 2. *Under the conditions of* a), e), g) *the equation* $L^*u=0$[2] *adjoint to* $Lu=0$ *has the fundamental solution* $Z^*(t,\eta,x,\xi)$

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1) Use the following inequality.
For

$$I(x,\xi,t,\tau)=\int_\tau^t d\beta\int_{-\infty}^{\infty}\exp(-af(x,\xi,y,\beta))\frac{dy}{[(t-\beta)(\beta-\tau)]^{1/(2b)}},$$
$$f(x,\xi,y,\beta)=\frac{(x-y)^q}{(t-\beta)^{1/(2b-1)}}+\frac{(y-\xi)^q}{(\beta-\tau)^{1/(2b-1)}},\ a>0,\ -\infty<x,y<\infty ,$$

then

$$|I(x,\xi,t,\tau)|\leqq\frac{M(\varepsilon)}{(t-\tau)^{1/(2b)}}\exp(-a(1-\varepsilon)|x-\xi|^q(t-\tau)^{-1/(2b-1)}),\ 0<\varepsilon<1 .$$

([1]), § 2, 4, lemma 3)

2) L^*u is the following system,

$$-\frac{\partial u_i}{\partial t}-\sum_{j=1}^{N}\sum_{\Sigma k_s\leqq 2b}(-1)^{k_1+\cdots+k_n}\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}\{A_{ji}^{(k_1,\cdots,k_n)}(x_1,\cdots,x_n,t)u_j\}$$
$$(i=1,\cdots,N) .$$

$(t_0 \leqq t < \eta \leqq T)$ towards the past, which has the same properties as that of $Z(t, \tau, x, \xi)$.

Let $u(x, t)$ be a solution of (1) whose initial value is equal to zero, and satisfy $|D_x^m u(x, t)| \leqq c \exp(k \sum_1^n |x_s|^q)$ $(m=1, \cdots, 2b-1)$, where $t_0 \leqq t \leqq t_0 + T_1$, c, k are positive constants, $T_1 = \left(\frac{c^* - \mu}{k}\right)^{2b-1}$, μ is any positive constant $< c^*$, then $u(x, t) \equiv 0$

PROOF. The first part is obtained directly. To prove the second part, we integrate $Z^*(t, t', x, x_0)Lu(x, t)$ $(t_0 \leqq t \leqq t'' < t' \leqq t_0 + T_1)$ in the cylinder $B_R\{t_0 \leqq t \leqq t'', x_i^0 - R \leqq x_i \leqq x_i^0 + R, i=1, \cdots, n, (x_1^0, \cdots, x_n^0) = x_0\}$.

By Green's formula we have $0 = \int_{V_R} u(x, t'')Z^*(t'', t', x, x_0)dx + S$, $(V_R = \{x; |x_i - x_i^0| \leqq R, i=1, \cdots, n\})$, where S is the surface integral over the side of the cylinder, whose integrand don't involve $D_t u$, $D_x^{2b} u$, $D_t Z^*$, $D_x^{2b} Z^*$. Let $R \to \infty$, then $S \to 0$, and the first integral tends to $\int u(x, t'')Z^*(t'', t', x, x_0)dx$. We write the above integral as following two integrals.

$$\int u(x, t'')Z^*(t'', t', x, x_0)dx$$
$$= \int G^*(t'', t', x - x_0, x)u(x, t'')dx + \int W^*(t'', t', x, x_0)u(x, t'')dx .$$

Let $t'' \to t'$, then the second integral tends to zero (use theorem 1), and the first integral tends to $u(x_0, t')$. Thus we proved the theorem 2. More detailed properties of the fundamental solution are stated in [3], 2, 6, [1], §2, 5 and 6.

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